

第6章

矩阵的Kroneker积和Hadamard积

The Kroneker Product and Hadamard Product

概述：

内容：

- 介绍Kronecker积和Hadamard积
- 讨论
 - K-积，H-积的运算性质、之间的关系
 - K-积与矩阵乘积的关系
 - K-积，H-积的矩阵性质
 - K-积的矩阵等价与相似关系
- 介绍应用
 - 向量化算子
- 重点：K-积及其应用

6.1 Kroneker积和Hadamard积的定义

• 定义6.1 (P. 136)

— 设矩阵 $A=[a_{ij}]_{m \times n}$ 和 $B=[b_{ij}]_{s \times t}$ 矩阵，则 A, B 的 Kronecher 被定义为 $A \otimes B$:

$$A \otimes B = [a_{ij}B]_{m \times n}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

$m \times s$
 $n \times t$

— 设 $A=[a_{ij}]_{m \times n}$ 和 $B=[b_{ij}]_{m \times n}$ 为同阶矩阵，则 A 和 B 的 Hadamard 被定义为 $A \circ B$:

$$A \circ B = [a_{ij}b_{ij}]_{m \times n}$$

例 设 $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ 计算

异同 异同

$A \otimes B$, $B \otimes A$, $I_2 \otimes B$, $A \circ B$, $I_2 \circ A$

• K-积, H-积的基本结果:

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) = I \otimes I = I$$

– A和B中有一个为零矩阵, 则 $A \otimes B = 0$, $A \circ B = 0$

– $| \otimes | = |$, $| \circ | = |$
 $n \times n \quad m \times m \quad n^2 \times n^2 \quad n \times m \quad m \times n \quad n \times n$

– 若A为对角矩阵, 则 $A \otimes B$ 为分块对角矩阵, $A \circ B$ 为对角矩阵。
 $(A \otimes B) \otimes C = [(A \otimes B) \otimes C]_{ij} = [A \otimes (B \otimes C)]_{ij}$

• K-积的基本性质

– 定理6.1 设以下矩阵使计算有意义, 则

- $(kA) \otimes B = A \otimes (kB)$
- $A \otimes (B+C) = A \otimes B + A \otimes C$
- ✓ • $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $(A \otimes B)^H = A^H \otimes B^H$ $(A \otimes B)^H = B^H \otimes A^H \times = A^H \otimes B^H$
- $A \otimes B \neq B \otimes A$
- 若方阵A, B可逆, 则 $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

$$(A \otimes B)^H = \begin{bmatrix} a_{11}^H B^H & a_{12}^H B^H \\ a_{21}^H & a_{22}^H B^H \end{bmatrix}$$

• K-积与矩阵乘法

— **定理6.2 (P. 138)** 设矩阵 A, B, C, D 使得下列运算有意义, 则有

$$\underbrace{(A \otimes B)}_F \underbrace{(C \otimes D)}_G = \underbrace{(AC)}_F \otimes \underbrace{(BD)}_G$$

— **意义**: 建立 Kronecher 积和矩阵乘法的相互转换。

— **特别情形**: 设 $A \in F^{m \times m}$, $B \in F^{n \times n}$, 则

$$\star A \otimes B = (I_m \otimes B) (A \otimes I_n) = (A \otimes I_n) (I_m \otimes B)$$

$$\begin{aligned} \tilde{F}_{ij} &= ((A \otimes B)(C \otimes D))_{ij} = \sum_{k=1}^n (A \otimes B)_{ik} \cdot (C \otimes D)_{kj} = \sum_{k=1}^n (a_{ik} B) (C_{kj} D) \\ &= \sum_{k=1}^n a_{ik} C_{kj} BD = \left(\sum_{k=1}^n a_{ik} C_{kj} \right) \otimes BD \end{aligned}$$

—— 数字

• H-积的基本性质:

设 A, B 为同阶矩阵, 则

$$- A \circ B = B \circ A$$

$$- (kA) \circ B = A \circ (kB)$$

$$- A \circ (B+C) = A \circ B + A \circ C$$

$$\checkmark - (A \circ B) \circ C = A \circ (B \circ C)$$

$$\checkmark - (A \circ B)^H = A^H \circ B^H$$

• Kronecher 和 Hadamard 的关系:

$$(A \otimes B)^H = B^H \cdot A^H \quad \left\{ \begin{array}{l} (A \otimes B)^H = A^H \otimes B^H \end{array} \right.$$

$$(A \otimes B)_{ij} = (a_{ij} b_{ij})^H = a_{ij}^H \cdot b_{ij}^H = (A^H \otimes B^H)_{ij}$$

— 例2:

$$\text{设 } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}, \text{ 计算 } A \otimes B, B \otimes A, A \circ B \text{ 和 } B \circ A.$$

$$A \otimes B = \begin{pmatrix} 1B & 2B \\ 3B & 4B \end{pmatrix} = \left[\begin{array}{cc|cc} -1 & 5 & 2 & 10 \\ 4 & -3 & 8 & -6 \\ \hline -3 & 15 & -4 & 20 \\ 12 & 9 & 16 & -12 \end{array} \right],$$

$$B \otimes A = \begin{pmatrix} -A & 5A \\ 4A & -3A \end{pmatrix} = \left[\begin{array}{cc|cc} -1 & -2 & 5 & 10 \\ -3 & -4 & 15 & 20 \\ \hline 4 & 8 & -3 & -6 \\ 12 & 16 & -9 & -12 \end{array} \right],$$

$$\checkmark A \circ B = \begin{pmatrix} -1 & 10 \\ 12 & -12 \end{pmatrix}, \quad B \circ A = \begin{pmatrix} -1 & 10 \\ 12 & -12 \end{pmatrix}.$$

对比 $A \circ B$ 与 $A \otimes B$ 的第一行第一列、第四行第四列交叉的元素！

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

• Kronecher和Hadamard的关系: $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i\text{th}$

— 定理6.3 (P. 139) $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

定理 6.3 设 $A, B \in F^{n \times n}$, 集合 $S = \{1, n+2, 2n+3, 3n+4, \dots, (n-1)n + n = n^2\}$, 则 Hadamard 积 $A \circ B$ 是在 Kronecker 积 $A \otimes B$ 中同时取 S 中的数对应的行和列得到的子矩阵, 记为 $A \otimes B \{S\} = A \circ B$.

$$a_{ij} = e_i^T A e_j \quad , \quad b_{ij} = e_i^T B e_j$$

$|i \times n \quad n \times n \quad n \times 1|$

$$e_1 \otimes e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n^2 \times 1}$$

第 1 行 第 1 列

$$e_2 \otimes e_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{n^2 \times 1}$$

第 $n+2$ 行

$$a_{ij} b_{ij} = (e_i^T A e_j) \cdot (e_i^T B e_j)$$

$$= (e_i^T A e_j) \otimes (e_i^T B e_j)$$

$$= (e_i^T \otimes e_i^T) (A \otimes B) (e_j \otimes e_j)$$

$$= \underline{(e_i \otimes e_i)^T} (A \otimes B) \underline{(e_j \otimes e_j)}$$

$$e_3 \otimes e_3 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{n^2 \times 1} = \begin{bmatrix} 0 \cdot e_3 \\ 0 \cdot e_3 \\ \vdots \\ 1 \cdot e_3 \\ \vdots \\ 0 \cdot e_3 \end{bmatrix}$$

第 $2n+3$ 行

$$\bar{E} = (e_1 \otimes e_1, e_2 \otimes e_2, \dots, e_n \otimes e_n)_{n^2 \times n}$$

$$\sqrt{A \circ B} = \begin{bmatrix} a_{11} b_{11} & a_{12} b_{12} & \dots \\ a_{21} b_{21} & \dots & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \underbrace{\bar{E}^T}_{n^2 \times n} \cdot \underbrace{(A \otimes B)}_{n^2 \times n^2} \cdot \underbrace{\bar{E}}_{n^2 \times n} = n \times n$$

$$\frac{(e_i \otimes e_i)^T}{(e_j \otimes e_j)}$$

6.2 Kronecher积和Hadamard积的性质

• Kronecher积的矩阵性质

— **定理6.4** 设矩阵使下列运算有意义，则

- 当A, B分别为可逆矩阵时, $A \otimes B$ 为可逆矩阵, 而且有

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

- 当方阵 $A \in F^{m \times m}$, $B \in F^{n \times n}$ 时, 方阵 $A \otimes B \in F^{mn \times mn}$ 的行
列式为

$$A \otimes B = (1A) \otimes (BI) = (I \otimes B)(A \otimes I)$$

$\downarrow_{m \times m} \quad \downarrow_{n \times n}$

$$|A \otimes B| = |A|^n |B|^m$$

$$|A \otimes B| = |I \otimes B| \cdot |A \otimes I|$$

- 若A, B是Hermite矩阵, 则 $A \otimes B$ 是Hermite矩阵

- 若A, B是酉矩阵, 则 $A \otimes B$ 是酉矩阵。

$$J_A \otimes I_{n \times n} = \begin{bmatrix} J_A & & \\ & \ddots & \\ & & J_A \end{bmatrix}$$

$$|J_A|^n, |A|^n$$

$$A = P J_A P^{-1} \quad I = P I P^{-1}$$

$$A \otimes I = \begin{bmatrix} A & & \\ & A & \\ & & \ddots \\ & & & A \end{bmatrix} \quad \times$$

$$A \otimes I = (P J_A P^{-1}) \otimes (P I P^{-1})$$

$$= (P \otimes P)(J_A \otimes I)(P^{-1} \otimes P^{-1}) = (P \otimes P)(J_A \otimes I)(P \otimes P)^{-1}$$

• Kronecher与矩阵等价、相似关系

定理6.5

- ◆ 设矩阵A, B, 为同阶的等价矩阵, 则 $(A \otimes I)$ 等价于 $(B \otimes I)$
- ◆ 设方阵A相似与 J_A , 方阵B相似于 J_B , 则 $(A \otimes B)$ 相似于 $(J_A \otimes J_B)$

$$A = P J_A P^{-1} \quad B = Q J_B Q^{-1}$$

$$A \otimes B = (P J_A P^{-1}) \otimes (Q J_B Q^{-1})$$

$$= (P \otimes Q) (J_A \otimes J_B) (P^{-1} \otimes Q^{-1})$$

$$= \underbrace{(P \otimes Q)} (J_A \otimes J_B) \underbrace{(P^{-1} \otimes Q^{-1})}^{-1}$$

• K-积特征值和特征向量

定理6.6 设 $A \in F^{m \times m}$ 的特征值特征向量分别是 λ_i, x_i , $B \in F^{n \times n}$ 的特征值、特征向量分别是 μ_j, y_j , 则

✓ — $(A \otimes B)$ 的特征值是 $\lambda_i \mu_j$ 。特征向量是 $(x_i \otimes y_j)$ 。

— $(A \otimes I) + (I \otimes B)$ 的特征值是 $\lambda_i + \mu_j$, 特征向量是 $(x_i \otimes y_j)$

$$\textcircled{1} \quad \underline{(A \otimes B)(x_i \otimes y_j)} = (Ax_i) \otimes (By_j) = \lambda_i x_i \otimes (\mu_j y_j) = \lambda_i \mu_j (x_i \otimes y_j)$$

$$\begin{aligned} \textcircled{2} \quad (A \otimes I + I \otimes B)(x_i \otimes y_j) &= (A \otimes I)(x_i \otimes y_j) + (I \otimes B)(x_i \otimes y_j) \\ &= (Ax_i) \otimes y_j + x_i \otimes (By_j) \\ &= (\lambda_i + \mu_j) \cdot (x_i \otimes y_j) \end{aligned}$$

◆ 更一般的结果:

定理6.7 的特征值为

$$P(A, B) = \sum_{i,j=0}^T c_{ij} A^i \otimes B^j$$

$$P(\lambda_r, \mu_t) = \sum_{i,j=0}^T c_{ij} \lambda_r^i \mu_t^j$$

- Kronecher积的矩阵函数性质

- 定理6.8 (P. 143) 设是 $f(z)$ 解析函数, $f(A)$ 有意义, 则

- $f(I \otimes A) = I \otimes f(A)$

- $f(A \otimes I) = f(A) \otimes I$

$$f(z) = \sum_{k=0}^N c_k z^k$$

$$f(I \otimes A) = \sum_{k=0}^N c_k (I \otimes A)^k$$

$$\Rightarrow = I \otimes \left(\sum_{k=0}^N c_k A^k \right) ?$$

- Kronecher积的矩阵函数性质
- 定理6.8 (P. 143) 设是 $f(z)$ 解析函数, $f(A)$ 有意义, 则

$$- f(I \otimes A) = I \otimes f(A)$$

$$- f(A \otimes I) = f(A) \otimes I$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \dots$$

- 特例:

$$- e^{I_m \otimes A} = I_m \otimes e^A$$

$$- e^{A \otimes I_m} = e^A \otimes I_m$$

- 例 设 $A \in F^{m \times n}$, $B \in F^{s \times t}$, 证明
 $\text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B)$

✱例 设 $A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

- ◆求 $(A \otimes B)$ 的特征值和特征向量
- ◆求 $[(A \otimes I) + (I \otimes B)]$ 的特征值和特征向量

例题3: 证明对任何方阵, 有

$$e^{A \oplus B} = e^A \otimes e^B = e^B \otimes e^A$$

6.3 矩阵的向量化算子和K-积

- 向量化算子 **Vec**

- 定义: 设 $A=[a_{ij}]_{m \times n}$ 则

$$\text{Vec}(A) = (a_{11} \ a_{21} \ \dots \ a_{m1}; \ a_{12} \ a_{22} \ \dots \ a_{m2}; \ \dots; \ a_{1n} \ a_{2n} \ \dots \ a_{mj})^T$$

- 性质:

1. Vec是线性算子:

$$\text{Vec}(\underline{k_1 A + k_2 B}) = \underline{k_1 \text{Vec}(A) + k_2 \text{Vec}(B)}$$

- 2 定理6.10 (P. 146) $\text{Vec}(ABC) = (C^T \otimes I) \text{Vec}B$

- 3 $\text{Vec}(AX) = (I \otimes A) \text{Vec}X$

- 4 $\text{Vec}(XC) = (C^T \otimes I) \text{Vec}X$

求证: $\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B)$

$B = (B_1, B_2, \dots, B_s)$, $C = (C_1, C_2, \dots, C_n)$

$$ABC = \underbrace{A \cdot (B_1, B_2, \dots, B_s)}_{\textcircled{1}} \underbrace{C}_{\textcircled{2}} = A \left(\sum_{j=1}^s C_{1j} B_j, \sum_{j=1}^s C_{2j} B_j, \dots, \sum_{j=1}^s C_{nj} B_j \right) \Rightarrow \text{vec}$$

$$= \left(\underbrace{A \cdot \sum_{j=1}^s C_{1j} B_j}_{\textcircled{3}}, A \sum_{j=1}^s C_{2j} B_j, \dots, A \sum_{j=1}^s C_{nj} B_j \right)$$

$$= \left(\underbrace{A \cdot [C_{11} \ C_{12} \ \dots \ C_{1s}]}_{\textcircled{4}} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix}, A [C_{21} \ C_{22} \ \dots \ C_{2s}] \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix}, \dots, A [C_{n1} \ C_{n2} \ \dots \ C_{ns}] \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix} \right)$$

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix} = \text{vec}(B) \quad \left(A \cdot \begin{bmatrix} \text{vec}(B) \end{bmatrix}, A \begin{bmatrix} \text{vec}(B) \end{bmatrix}, \dots, A \begin{bmatrix} \text{vec}(B) \end{bmatrix} \right)$$

$$(A C_1^T, A C_2^T, \dots, A C_n^T) \text{vec}(B)$$

$$\text{vec}(ABC) = \text{vec}(A C_1^T \text{vec}(B), A C_2^T \text{vec}(B), \dots, A C_n^T \text{vec}(B))$$

↓

$$\text{vec} \begin{bmatrix} A C_1^T \text{vec}(B) \\ \vdots \\ A C_n^T \text{vec}(B) \end{bmatrix} = \begin{bmatrix} C_{11}A & C_{21}A & \dots \\ C_{12}A & C_{22}A & \dots \\ \vdots & \vdots & \ddots \\ C_{1n}A & \vdots & \ddots \end{bmatrix} \cdot \text{vec}(B) = (C^T \otimes A) \text{vec}(B)$$

证毕

用向量化算子求解矩阵方程

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B)$$

思想：用Vec算子，结合Kronecher积将矩阵方程化为线性方程组求解。

$$\underline{Ax = b}$$

1、 $A \in \mathbb{F}^{m \times m}$, $B \in \mathbb{F}^{n \times n}$, $D \in \mathbb{F}^{m \times n}$, $\mathbf{AX+XB=D}$



分析：

$$Ax + xB = D$$

$$AX+XB=D \Leftrightarrow (I \otimes A + B^T \otimes I) \text{Vec}X = \text{Vec}D$$

$$\bullet \quad G = (I \otimes A + B^T \otimes I), \quad = I \otimes A \text{vec}(X) + B^T \otimes I \text{vec}(X)$$

• 方程有惟一解的充要条件是 G 为可逆矩阵，即 A 和 $-B$ 没有共同的特征值。

用向量化算子求解矩阵方程

- 2、 $A, X \in F^{n \times n}$, $AX - XA = kX$

$$\text{vec}(AX - XA) = \text{vec}(kX)$$

$$\text{vec}(AX) - \text{vec}(XA) = \text{vec}(kX)$$

$$[I \otimes A] \text{vec}(X) - [A^T \otimes I] \text{vec}(X) = k \text{vec}(X)$$

$$([I \otimes A - A^T \otimes I] - kI) \text{vec}(X) = 0$$

分析:

$$AX - XA = kX \Leftrightarrow (I \otimes A - A^T \otimes I) \text{vec}X = k \text{vec}X$$

- $H = (I \otimes A - A^T \otimes I)$,
- 方程 $(kI - H)y = 0$ 有非零解的充要条件是 k 为 H 的特征值, $k = \lambda_i - \lambda_j$ 。

例题2 求解矩阵方程 $AX - XA = -2X$, $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

用向量化算子求解矩阵方程

3. $A, B, D, X \in F^{n \times n}$, $AXB=D$

• 分析: $AXB=D \Leftrightarrow (B^T \otimes A) \text{Vec}X = \text{Vec}D$

✧ $L = B^T \otimes A$,

✧ 方程有惟一解的充要条件是L为可逆矩阵. 例题3
求解方程 $A_1XB_1 + A_2XB_2 = D$

$$A_1 = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 4 & -6 \\ 0 & 8 \end{bmatrix}$$

例 设 $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{n \times n}$, $D \in \mathbb{F}^{m \times n}$, 证明谱半径 $\rho(A) \cdot \rho(B) < 1$ 时方程:

$$X = AXB + D$$

的解为

$$X = \sum_{k=0}^{\infty} A^k D B^k$$

$$\text{vec}(X) = \text{vec}(AXB) + \text{vec}(D)$$

$$(I - B^T \otimes A) \text{vec}(X) = \text{vec}(D)$$

$$\text{vec}(X) = \underbrace{(I - B^T \otimes A)^{-1}}_{\Downarrow} \text{vec}(D)$$

$$(I - A)^{-1} = I + A + A^2 + A^3 + \dots + A^n + \dots \quad \text{若 } A \text{ 的谱半径小于 } 1 \text{ (矩阵幂级数)}$$

$$\begin{aligned} \Downarrow \quad \text{vec}(X) &= (I + B^T \otimes A + (B^T \otimes A)^2 + \dots)^{\text{vec}(D)} = D + B^T \otimes AD + \dots \\ &= D + ADB + A^2 D B^2 + \dots = \sum_{k=0}^{\infty} A^k D B^k \end{aligned}$$